

# Homogeneous approximation property for continuous wavelet transforms<sup>☆</sup>

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## Abstract

The homogeneous approximation property (HAP) for frames is useful in practice and has been developed recently. In this paper, we study the HAP for the continuous wavelet transform. We show that every pair of admissible wavelets possesses the HAP in  $L^2$  sense, while it is not true in general whenever pointwise convergence is considered. We give necessary and sufficient conditions for the pointwise HAP to hold, which depends on both wavelets and functions to be reconstructed.

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## 1. Introduction

The wavelet transform of a function  $f \in L^2(\mathbb{R})$  with respect to  $\psi \in L^2(\mathbb{R})$  is defined by

$$\langle f, \tau(a, b)\psi \rangle = |a|^{-1/2} \int_{-\infty}^{+\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx,$$

where

$$(\tau(a, b)\psi)(x) = |a|^{-1/2} \psi(a^{-1}(x - b)).$$

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The wavelet transform is a useful tool that cuts up a function  $f$  into different frequency components and then studies properties of  $f$  with each component.

There exist many different types of wavelet transforms, all starting from the following two cases:

- (i). The continuous wavelet transform  $\langle f, \tau(a, b)\psi \rangle$  where the dilation and translation parameters  $a, b$  vary continuously over  $\mathbb{R}$  with the constraint  $a \neq 0$ .
- (ii). The discrete wavelet transform  $\langle f, \tau(a, b)\psi \rangle$  where the dilation and translation parameters  $a, b$  both take only discrete values.

In the later case, the corresponding  $\tau(a, b)\psi$  usually forms a frame for  $L^2(\mathbb{R})$ , called a wavelet frame. The theory of wavelet frames has been developed very fast over the last twenty years and it plays an important role in modern time-frequency analysis. Various properties of wavelet frames can be seen in [1–8, 13, 16]. See also [11, 18–20] for some studies on irregular wavelet frames.

Recently, the homogeneous approximation property (HAP) for wavelet frames has been studied in [1, 9, 10, 12, 15, 17]. It was shown that every wavelet frame with nice generators possesses the HAP, which is useful in practice since it means that the number of building blocks involved in a reconstruction of  $f$  up to some error is essentially invariant under time-scale shifts.

In this paper, we study the homogeneous approximation property for the continuous wavelet transform. First, we introduce some notations.

The group action in  $\mathcal{G} = \{(a, b) : a, b \in \mathbb{R}, a \neq 0\}$  is defined by

$$(a, b)(s, t) = (as, b + at).$$

The unit element is  $(1, 0)$ , and the inverse of  $(a, b) \in \mathcal{G}$  is  $(1/a, -b/a)$ .

For  $A_2 > A_1 > 0$  and  $B > 0$ , denote

$$Q_{A_1, A_2; B} = ([-A_2, -A_1] \cup [A_1, A_2]) \times [-B, B].$$

For every  $(s, t) \in \mathcal{G}$ , its  $(A_1, A_2; B)$ -neighborhood is defined by

$$(s, t)Q_{A_1, A_2; B} = \{(sa, t + bs) : a \in [-A_2, -A_1] \cup [A_1, A_2], b \in [-B, B]\}.$$

We call a function  $\psi \in L^2(\mathbb{R})$  admissible if

$$C_\psi = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega < +\infty.$$

For admissible  $\psi_1, \psi_2$ , denote

$$C_{\psi_1, \psi_2} = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} \overline{\hat{\psi}_1(\omega)} \hat{\psi}_2(\omega) d\omega.$$

Whenever  $C_{\psi_1, \psi_2} \neq 0$ , we see from the wavelet theory [5, page 28] that

$$f(x) = C_{\psi_1, \psi_2}^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle f, \tau(a, b)\psi_1 \rangle \langle \tau(a, b)\psi_2 \rangle(x) \frac{1}{a^2} da db, \quad (1.1)$$

where the convergence is both in  $L^2(\mathbb{R})$  and pointwise in some cases [5, Theorem 2.4.2] (see Proposition 2.2).

**Definition 1.1.** A pair of admissible wavelets  $(\psi_1, \psi_2)$  is said to possess the homogeneous approximation property in  $L^2(\mathbb{R})$  if for any  $f \in L^2(\mathbb{R})$  and  $\varepsilon > 0$ , there exist some  $A_2 > A_1 > 0$

and  $B > 0$  such that

$$\left\| \tau(s, t)f - C_{\psi_1, \psi_2}^{-1} \iint_{(a, b) \in (s, t) Q_{A'_1, A'_2, B'}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \tau(a, b)\psi_2 \frac{1}{a^2} da db \right\|_2 \leq \varepsilon, \\ \forall (s, t) \in \mathcal{G}, A'_2 \geq A_2, 0 < A'_1 \leq A_1, B' \geq B. \quad (1.2)$$

In this paper, we show that any pair of wavelets  $(\psi_1, \psi_2)$  with  $C_{\psi_1, \psi_2} \neq 0$  possesses the homogeneous approximation property in  $L^2(\mathbb{R})$ .

On the other hand, it was shown in [5, 14] that for certain  $\psi_1, \psi_2$  and  $f$ , in every point  $x$  where  $f$  is continuous, we have

$$f(x) = \lim_{\substack{A_1 \rightarrow 0 \\ A_2 \rightarrow \infty}} C_{\psi_1, \psi_2}^{-1} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle \langle \tau(a, b)\psi_2 \rangle(x) db. \quad (1.3)$$

In other words, the integral in (1.1) also converges pointwise. For this case, we can also consider the homogeneous approximation property.

**Definition 1.2.** A pair of admissible wavelets  $(\psi_1, \psi_2)$  is said to possess the pointwise homogeneous approximation property if for any  $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ ,  $x \in \mathbb{R}$  and  $\varepsilon > 0$ , there exist some  $A_2 > A_1 > 0$  such that

$$\left| (\tau(s, t)f)(x) - C_{\psi_1, \psi_2}^{-1} \int_{A'_1 |s| \leq |a| \leq A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle \langle \tau(a, b)\psi_2 \rangle(x) db \right| \\ \leq \varepsilon, \quad \forall A'_2 \geq A_2, 0 < A'_1 \leq A_1, (s, t) \in \mathcal{G}. \quad (1.4)$$

Does every pair of admissible wavelets possess the pointwise HAP? The answer is negative, even if all of  $\psi_1, \psi_2$  and  $f$  are Schwartz functions. The following is a counterexample.

**Example 1.1.** Let

$$\hat{\psi}(\omega) = \hat{\psi}_1(\omega) = \hat{\psi}_2(\omega) = e^{-\omega^2} - e^{-2\omega^2}, \quad \omega \in \mathbb{R}.$$

For any bounded continuous function  $f \in L^2(\mathbb{R})$  with  $\|f\|_2 \neq 0$ ,  $A_2 > A_1 > 0$ ,  $x \in \mathbb{R}$ , and  $\eta > 0$ , there is some  $(s, t) \in \mathcal{G}$  such that

$$\left| (\tau(s, t)f)(x) - C_{\psi}^{-1} \int_{A_1 |s| \leq |a| \leq A_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi \rangle \langle \tau(a, b)\psi \rangle(x) db \right| \\ = \eta.$$

A natural problem arises: does the HAP hold for the pointwise convergence in some sense? We show that it is the case whenever  $(s, t)$  is chosen such that  $|s| > s_0$  for some  $s_0 > 0$ . Moreover, we give necessary and sufficient conditions under which (1.4) holds. In particular, we show that if (1.4) holds for some  $x \in \mathbb{R}$ , then it is true for every  $x \in \mathbb{R}$ . Furthermore, there exist some  $A_2 > A_1 > 0$  such that

$$(\tau(s, t)f)(x) = C_{\psi_1, \psi_2}^{-1} \int_{A'_1 |s| \leq |a| \leq A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \\ \times \langle \tau(a, b)\psi_2 \rangle(x) db, \quad \forall A'_2 \geq A_2, 0 < A'_1 \leq A_1, (s, t) \in \mathcal{G}.$$

In other words, we can choose  $\varepsilon$  to be 0.

The paper is organized as follows. We collect some preliminary results in Section 2. In Section 3, we show the HAP for a pair of wavelets in  $L^2$ -norm and  $L^\infty$ -norm respectively. Moreover, we give necessary and sufficient conditions under which (1.4) holds. At last, we give an example which satisfies (1.4).

## 2. Notations and preliminary results

The Fourier transform of  $f \in L^2(\mathbb{R})$  is defined by  $\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}dx$ .

For  $b > a > 0$ , define

$$AC_{a,b}(\mathbb{R}) = \{f : \text{supp } f \subset [-b, -a] \cup [a, b]\}.$$

Let

$$AC(\mathbb{R}) = \bigcup_{0 < a < b < +\infty} AC_{a,b}(\mathbb{R}).$$

It is easy to see that  $AC(\mathbb{R})$  contains all functions which are compactly supported on  $\mathbb{R} \setminus \{0\}$ .

For  $\psi_1, \psi_2 \in L^2(\mathbb{R})$ , define

$$P_{\psi_1, \psi_2}(\omega) = \overline{\hat{\psi}_1(\omega)}\hat{\psi}_2(\omega) + \overline{\hat{\psi}_1(-\omega)}\hat{\psi}_2(-\omega), \quad \forall \omega \in \mathbb{R}.$$

Next we introduce some preliminary results. We begin with an identity for the continuous wavelet transform.

**Proposition 2.1** ([5, Proposition 2.4.1]). For all  $f, g \in L^2(\mathbb{R})$ , we have

$$\iint_G \langle f, \tau(a, b)\psi_1 \rangle \overline{\langle g, \tau(a, b)\psi_2 \rangle} \frac{1}{a^2} da db = C_{\psi_1, \psi_2} \langle f, g \rangle. \quad (2.1)$$

The following result shows that we can reconstruct a function pointwise in some cases.

**Proposition 2.2** ([5, Proposition 2.4.2]). Suppose that  $\psi_1, \psi_2 \in L^1(\mathbb{R})$ , that  $\psi_2$  is differentiable with  $\psi_2' \in L^2(\mathbb{R})$ , that  $x\psi_2 \in L^1(\mathbb{R})$ , and that  $\hat{\psi}_1(0) = 0 = \hat{\psi}_2(0)$ . If  $f \in L^2(\mathbb{R})$  is bounded, then in every point  $x$  where  $f$  is continuous, we have

$$f(x) = C_{\psi_1, \psi_2}^{-1} \lim_{\substack{A_1 \rightarrow 0 \\ A_2 \rightarrow \infty}} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi \rangle \overline{\langle \tau(a, b)\psi, x \rangle} db. \quad (2.2)$$

For the decay of the wavelet transform, we have the following result.

**Proposition 2.3** ([5, Theorem 2.9.1]). Suppose that  $\int_{-\infty}^{+\infty} (1 + |x|)|\psi(x)|dx < +\infty$  and  $\hat{\psi}(0) = 0$ . If a bounded function  $f$  is Hölder continuous with exponent  $\alpha$ ,  $0 < \alpha \leq 1$ , i.e.,

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

then its wavelet transform satisfies

$$|\langle f, \tau(a, b)\psi \rangle| \leq C'|a|^{\alpha+1/2},$$

where  $C$  and  $C'$  are positive constants.

The following lemma gives a class of admissible wavelets.

**Lemma 2.4.** If  $\hat{\psi}(0) = 0$  and  $\int_{-\infty}^{+\infty} (1 + |x|)^\alpha |\psi(x)|dx < +\infty$  for some  $\alpha > 0$ , then  $C_\psi < +\infty$ .

**Proof.** It is easy to see that  $\psi \in L^1(\mathbb{R})$ . We only need to prove the result for the case of  $0 < \alpha \leq 1$ . Note that

$$C_\psi = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega = \int_{|\omega|>1} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega + \int_{-1}^1 \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega.$$

For the first part, we have

$$\int_{|\omega|>1} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega \leq \int_{-\infty}^{+\infty} |\hat{\psi}(\omega)|^2 d\omega < +\infty. \quad (2.3)$$

For the second part, since

$$\begin{aligned} |\hat{\psi}(\omega)| &= |\hat{\psi}(\omega) - \hat{\psi}(0)| = \left| \int_{-\infty}^{+\infty} \psi(x) e^{-ix\omega} dx - \int_{-\infty}^{+\infty} \psi(x) dx \right| \\ &\leq \int_{-\infty}^{+\infty} |\psi(x)| \cdot |e^{-ix\omega} - 1| dx = 2 \int_{-\infty}^{+\infty} \left| \psi(x) \sin \frac{x\omega}{2} \right| dx \\ &\leq 2 \int_{-\infty}^{+\infty} |\psi(x)| \cdot \left| \sin \frac{x\omega}{2} \right|^\alpha dx \leq 2^{1-\alpha} |\omega|^\alpha \int_{-\infty}^{+\infty} |x|^\alpha \cdot |\psi(x)| dx \\ &\leq M |\omega|^\alpha, \end{aligned}$$

we have

$$\int_{-1}^1 \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega \leq \int_{-1}^1 \frac{M^2}{|\omega|^{1-2\alpha}} d\omega < +\infty. \quad (2.4)$$

Now the conclusion follows by combining (2.3) and (2.4).  $\square$

### 3. Homogeneous approximation property for the continuous wavelet transform

#### 3.1. Homogeneous approximation property in $L^2(\mathbb{R})$

In this subsection, we prove the HAP in  $L^2(\mathbb{R})$  for a pair of wavelets  $(\psi_1, \psi_2)$ .

**Theorem 3.1.** *If  $\psi_1, \psi_2 \in L^2(\mathbb{R})$  are admissible and  $C_{\psi_1, \psi_2} \neq 0$ , then  $(\psi_1, \psi_2)$  possesses the homogeneous approximation property in  $L^2(\mathbb{R})$ .*

**Proof.** Let  $A_2 > A_1 > 0$  and  $B > 0$  be constants to be determined later. Suppose that  $A'_2 > A_2$  and  $0 < A'_1 \leq A_1$ . Then for any  $f \in L^2(\mathbb{R})$  and  $(s, t) \in \mathcal{G}$ , we have

$$\begin{aligned} &\left\| \tau(s, t)f - C_{\psi_1, \psi_2}^{-1} \iint_{(a, b) \in (s, t) \mathcal{Q}_{A'_1, A'_2; B'}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \tau(a, b)\psi_2 \frac{1}{a^2} da db \right\|_2^2 \\ &= \sup_{\|g\|_2=1} \left| \left\langle \tau(s, t)f - C_{\psi_1, \psi_2}^{-1} \iint_{(a, b) \in (s, t) \mathcal{Q}_{A'_1, A'_2; B'}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \right. \right. \\ &\quad \left. \left. \times \tau(a, b)\psi_2 \frac{da db}{a^2}, g \right\rangle \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \sup_{\|g\|_2=1} \left| C_{\psi_1, \psi_2}^{-1} \iint_{(a,b) \notin (s,t) Q_{A'_1, A'_2; B'}} \langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle \langle \tau(a, b) \psi_2, g \rangle \frac{1}{a^2} da db \right|^2 \\
&\quad \text{(using Proposition 2.1)} \\
&\leq \sup_{\|g\|_2=1} \left| C_{\psi_1, \psi_2}^{-2} \iint_{(a,b) \notin (s,t) Q_{A'_1, A'_2; B'}} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle|^2 \frac{1}{a^2} da db \right. \\
&\quad \times \left. \iint_{\mathcal{G}} |\langle g, \tau(a, b) \psi_2 \rangle|^2 \frac{1}{a^2} da db \right| \\
&= C_{\psi_2} \left| C_{\psi_1, \psi_2}^{-2} \iint_{(a,b) \notin (s,t) Q_{A'_1, A'_2; B'}} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle|^2 \frac{1}{a^2} da db \right| \\
&= C_{\psi_2} \left| C_{\psi_1, \psi_2}^{-2} \iint_{(a,b) \notin Q_{A'_1, A'_2; B'}} |\langle f, \tau(a, b) \psi_1 \rangle|^2 \frac{1}{a^2} da db \right| \\
&\quad ((a, b) \rightarrow (as, t + bs)) \\
&\leq C_{\psi_2} \left| C_{\psi_1, \psi_2}^{-2} \iint_{(a,b) \notin Q_{A_1, A_2; B}} |\langle f, \tau(a, b) \psi_1 \rangle|^2 \frac{1}{a^2} da db \right| \\
&:= E_{A_1, A_2; B}.
\end{aligned}$$

By Proposition 2.1, we can make  $E_{A_1, A_2; B}$  arbitrarily small by choosing  $A_2$  and  $B$  large enough and  $A_1$  small enough. This completes the proof.  $\square$

### 3.2. Homogeneous approximation property in $L^\infty(\mathbb{R})$

In this subsection, we study the HAP in  $L^\infty(\mathbb{R})$ . We show that the pointwise HAP depends on both the wavelets and the function to be reconstructed, which is quite different from the case of  $L^2(\mathbb{R})$ .

First, we consider the HAP for general functions.

**Theorem 3.2.** Suppose that  $\psi_1, \psi_2, x\psi_1, x\psi_2 \in L^1(\mathbb{R})$ , that  $\psi_2$  is differentiable with  $\psi'_2 \in L^2(\mathbb{R})$ , and that  $\hat{\psi}_1(0) = 0 = \hat{\psi}_2(0)$ . If a bounded function  $f$  is Hölder continuous with exponent  $\alpha$ ,  $0 < \alpha \leq 1$ , then for any  $\varepsilon, s_0 > 0$ , there exist constants  $A_2 > A_1 > 0$  such that for any  $(s, t) \in \mathcal{G}$  with  $|s| \geq s_0$ ,  $x \in \mathbb{R}$ ,  $A'_2 \geq A_2$  and  $0 < A'_1 \leq A_1$ , we have

$$\begin{aligned}
&\left| (\tau(s, t) f)(x) - C_{\psi_1, \psi_2}^{-1} \int_{A'_1 |s| \leq |a| \leq A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle \right. \\
&\quad \times \left. (\tau(a, b) \psi_2)(x) db \right| \leq \varepsilon.
\end{aligned} \tag{3.1}$$

**Proof.** By Lemma 2.4, it is easy to see that  $\psi_1$  and  $\psi_2$  are admissible. Proposition 2.2 implies that

$$\begin{aligned}
&\left| (\tau(s, t) f)(x) - C_{\psi_1, \psi_2}^{-1} \int_{A'_1 |s| \leq |a| \leq A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle \right. \\
&\quad \times \left. (\tau(a, b) \psi_2)(x) db \right|
\end{aligned}$$

$$\begin{aligned} &\leq \left| C_{\psi_1, \psi_2}^{-1} \right| \int_{0 < |a| < A'_1 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db \\ &\quad + \left| C_{\psi_1, \psi_2}^{-1} \right| \int_{|a| > A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db. \end{aligned}$$

Since

$$\begin{aligned} &\int_{|a| > A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db \\ &\leq \left( \iint_G |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle|^2 \frac{1}{a^2} da db \right)^{1/2} \\ &\quad \times \left( \int_{|a| > A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |(\tau(a, b) \psi_2)(x)|^2 db \right)^{1/2} \\ &= \frac{\sqrt{2} C_{\psi_1}^{1/2} \|f\|_2 \|\psi_2\|_2}{(A'_2 |s|)^{1/2}} \leq \frac{\sqrt{2} C_{\psi_1}^{1/2} \|f\|_2 \|\psi_2\|_2}{(A_2 s_0)^{1/2}}, \end{aligned}$$

we see that when  $A_2 = 8C_{\psi_1} \|f\|_2^2 \|\psi_2\|_2^2 / (|C_{\psi_1, \psi_2}| s_0 \varepsilon^2)$ , for any  $A'_2 \geq A_2$ , we get

$$\left| C_{\psi_1, \psi_2}^{-1} \right| \int_{|a| > A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db \leq \frac{\varepsilon}{2}. \quad (3.2)$$

On the other hand, by Proposition 2.3, there exists some constant  $C > 0$  such that

$$|\langle f, \tau(a, b) \psi_1 \rangle| \leq C |a|^{\alpha+1/2}.$$

Hence

$$\begin{aligned} &\int_{0 < |a| < A'_1 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db \\ &= \int_{0 < |a| < A'_1} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle f, \tau(a, b) \psi_1 \rangle \tau(as, bs + t) \psi_2(x)| db \\ &\leq C |s|^{-1/2} \int_{0 < |a| < A'_1} da \int_{-\infty}^{+\infty} \frac{1}{|a|^{2-\alpha}} \left| \psi_2 \left( \frac{x - bs - t}{as} \right) \right| db \\ &= \frac{2C \|\psi_2\|_1 (A'_1)^\alpha}{\alpha |s|^{1/2}} \leq \frac{2C \|\psi_2\|_1 A_1^\alpha}{\alpha s_0^{1/2}}. \end{aligned}$$

Thus, when  $A_1 = \left( \varepsilon \alpha s_0^{1/2} |C_{\psi_1, \psi_2}| / (4C \|\psi_2\|_1) \right)^{1/\alpha}$ , for any  $0 < A'_1 \leq A_1$ , we have that

$$\left| C_{\psi_1, \psi_2}^{-1} \right| \int_{0 < |a| < A'_1 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} |\langle \tau(s, t) f, \tau(a, b) \psi_1 \rangle (\tau(a, b) \psi_2)(x)| db \leq \frac{\varepsilon}{2}. \quad (3.3)$$

Now the conclusion follows by combining (3.2) and (3.3).  $\square$

Recall that Example 1.1 shows that the condition  $|s| \geq s_0$  is necessary. Before giving a proof for the example, we introduce a simple lemma, which can be proved similarly to [5, Proposition 2.4.1].

**Lemma 3.3.** Let  $\psi_1, \psi_2$  be admissible with  $C_{\psi_1, \psi_2} \neq 0$ . For any  $f \in L^2(\mathbb{R})$  and  $A_2 > A_1 > 0$ , define

$$f_{A_1, A_2}(x) = C_{\psi_1, \psi_2}^{-1} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db. \quad (3.4)$$

Then we have

$$(f_{A_1, A_2})^\vee(\omega) = C_{\psi_1, \psi_2}^{-1} \hat{f}(\omega) \int_{A_1 \leq |a| \leq A_2} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) \frac{da}{|a|}. \quad (3.5)$$

**Proof of example 1.1.** Obviously  $\hat{\psi}(0) = \hat{\psi}_1(0) = \hat{\psi}_2(0) = 0$ , and

$$C_\psi = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} |e^{-\omega^2} - e^{-2\omega^2}|^2 d\omega < +\infty.$$

Moreover, for  $0 < A_1 < A_2 < \infty$ , we have

$$C_\psi - \int_{A_1 \leq |a| \leq A_2} \frac{1}{|a|} |\hat{\psi}(a\omega)|^2 da > 0, \quad \forall \omega \in \mathbb{R}.$$

It is easy to see that  $\psi(x)$  satisfies the conditions in Proposition 2.2. Hence for a bounded and continuous function  $f$  in  $L^2(\mathbb{R})$ , we have

$$f(x) = C_\psi^{-1} \lim_{\substack{A_1 \rightarrow 0 \\ A_2 \rightarrow \infty}} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi \rangle (\tau(a, b)\psi)(x) db, \quad \forall x \in \mathbb{R}.$$

For any  $A_2 > A_1 > 0$ , let  $f_{A_1, A_2}$  be defined by (3.4). Then for any  $(s, t) \in \mathcal{G}$ ,

$$\begin{aligned} & |s|^{-1/2} f_{A_1, A_2} \left( \frac{x-t}{s} \right) \\ &= C_\psi^{-1} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi \rangle (\tau(as, bs+t)\psi)(x) db \\ &= C_\psi^{-1} \int_{A_1 |s| \leq |a| \leq A_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi \rangle (\tau(a, b)\psi)(x) db. \end{aligned}$$

Now we see from Lemma 3.3 that whenever  $\|f\|_2 \neq 0$ ,

$$\langle f - f_{A_1, A_2}, f \rangle = \frac{1}{2\pi C_\psi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \left( C_\psi - \int_{A_1 \leq |a| \leq A_2} \frac{1}{|a|} |\hat{\psi}(a\omega)|^2 da \right) d\omega \neq 0.$$

Hence  $\|f - f_{A_1, A_2}\|_2 > 0$ . Since  $f$  is continuous, there exists some  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq f_{A_1, A_2}(x_0)$ . Put  $|s|^{-1/2} = \eta |f(x_0) - f_{A_1, A_2}(x_0)|^{-1}$  and  $t = x - x_0 s$ . Then  $x_0 = (x-t)/s$ . Hence

$$\begin{aligned} & \left| (\tau(s, t)f)(x) - C_\psi^{-1} \int_{A_1 |s| \leq |a| \leq A_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi \rangle (\tau(a, b)\psi)(x) db \right| \\ &= |s|^{-1/2} \left| f \left( \frac{x-t}{s} \right) - f_{A_1, A_2} \left( \frac{x-t}{s} \right) \right| \\ &= |s|^{-1/2} |f(x_0) - f_{A_1, A_2}(x_0)| = \eta. \end{aligned}$$

This completes the proof.  $\square$



Next we give necessary and sufficient conditions under which (1.4) holds. In particular, we show that if (1.4) holds for some  $x \in \mathbb{R}$ , then it holds uniformly for every  $x \in \mathbb{R}$ . Moreover, it is even true for  $\varepsilon = 0$ .

**Theorem 3.4.** Suppose that  $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$  is continuous, that  $\psi_1, \psi_2 \in L^1(\mathbb{R})$ , that  $\psi_2$  is differentiable with  $\psi_2' \in L^2(\mathbb{R})$ , that  $x\psi_2 \in L^1(\mathbb{R})$ , and that  $\hat{\psi}_1(0) = 0 = \hat{\psi}_2(0)$ . Then the following assertions are equivalent.

(i) There is some  $x_0 \in \mathbb{R}$  such that for any  $\varepsilon > 0$ , there exist  $A_2 > A_1 > 0$  such that

$$\left| (\tau(s, t)f)(x_0) - C_{\psi_1, \psi_2}^{-1} \iint_{\substack{A_1' |s| \leq |a| \leq A_2' |s| \\ b \in \mathbb{R}}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \right. \\ \left. \times (\tau(a, b)\psi_2)(x_0) \frac{da db}{a^2} \right| \leq \varepsilon, \quad \forall (s, t) \in \mathcal{G}, A_2' \geq A_2, 0 < A_1' \leq A_1. \quad (3.6)$$

(ii) For any  $\varepsilon > 0$ , there exist  $A_2 > A_1 > 0$  such that

$$\left| (\tau(s, t)f)(x) - C_{\psi_1, \psi_2}^{-1} \iint_{\substack{A_1' |s| \leq |a| \leq A_2' |s| \\ b \in \mathbb{R}}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) \frac{da db}{a^2} \right| \\ \leq \varepsilon, \quad \forall (s, t) \in \mathcal{G}, x \in \mathbb{R}, A_2' \geq A_2, 0 < A_1' \leq A_1. \quad (3.7)$$

(iii) There exist  $A_2 > A_1 > 0$  such that

$$(\tau(s, t)f)(x) = C_{\psi_1, \psi_2}^{-1} \iint_{\substack{A_1' |s| \leq |a| \leq A_2' |s| \\ b \in \mathbb{R}}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle \\ \times (\tau(a, b)\psi_2)(x) \frac{da db}{a^2}, \quad \forall (s, t) \in \mathcal{G}, x \in \mathbb{R}, A_2' \geq A_2, 0 < A_1' \leq A_1. \quad (3.8)$$

(iv) There exist  $A_2 > A_1 > 0$  such that

$$f(x) = C_{\psi_1, \psi_2}^{-1} \int_{A_1' \leq |a| \leq A_2'} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db, \\ \forall x \in \mathbb{R}, A_2' \geq A_2, 0 < A_1' \leq A_1. \quad (3.9)$$

(v) There exist  $A_2 > A_1 > 0$  such that

$$C_{\psi_1, \psi_2} = \int_{A_1' \leq |a| \leq A_2'} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da, \\ \forall \omega \in \text{supp } \hat{f}, A_2' \geq A_2, 0 < A_1' \leq A_1. \quad (3.10)$$

(vi)  $\hat{f} \in AC(\mathbb{R})$  and  $P_{\psi_1, \psi_2} \in AC(\mathbb{R})$ .

**Proof.** For any  $A_2 > A_1 > 0$ , write

$$f_{A_1, A_2}(x) = C_{\psi_1, \psi_2}^{-1} \int_{A_1 \leq |a| \leq A_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db.$$

Then for any  $(s, t) \in \mathcal{G}$ , we have

$$(\tau(s, t)f_{A_1, A_2})(x) = |s|^{-1/2} f_{A_1, A_2} \left( \frac{x-t}{s} \right) \\ = C_{\psi_1, \psi_2}^{-1} \int_{A_1 |s| \leq |a| \leq A_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db. \quad (3.11)$$

(i) $\implies$ (ii). Assume that (i) holds. By substituting  $t + x_0 - x$  for  $t$  in (3.6), we get

$$\left| (\tau(s, t)f)(x) - C_{\psi_1, \psi_2}^{-1} \iint_{\substack{A'_1 |s| \leq |a| \leq A'_2 |s| \\ b \in \mathbb{R}}} \langle \tau(s, t + x_0 - x)f, \tau(a, b)\psi_1 \rangle \right. \\ \left. \times (\tau(a, b)\psi_2)(x_0) \frac{da db}{a^2} \right| \leq \varepsilon.$$

On the other hand, by a change of variable of the form  $b \rightarrow b + x_0 - x$ , we get

$$\iint_{\substack{A'_1 |s| \leq |a| \leq A'_2 |s| \\ b \in \mathbb{R}}} \langle \tau(s, t + x_0 - x)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x_0) \frac{da db}{a^2} \\ = \iint_{\substack{A'_1 |s| \leq |a| \leq A'_2 |s| \\ b \in \mathbb{R}}} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) \frac{da db}{a^2}.$$

Hence (ii) holds.

(ii) $\implies$ (iv). Fix some  $\varepsilon > 0$ . Suppose that (3.7) holds for some  $A_2 > A_1 > 0$ . By choosing  $t = x - xs$ , we see from (3.11) that

$$\left| f(x) - f_{A'_1, A'_2}(x) \right| \\ = |s|^{1/2} \left| (\tau(s, t)f)(x) - \left( \tau(s, t)f_{A'_1, A'_2} \right)(x) \right| \\ = |s|^{1/2} \left| (\tau(s, t)f)(x) - C_{\psi_1, \psi_2}^{-1} \right. \\ \left. \times \int_{A'_1 |s| \leq |a| \leq A'_2 |s|} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db \right| \\ < \varepsilon |s|^{1/2}. \quad (3.12)$$

Since  $s$  is arbitrary, we conclude that

$$f(x) = C_{\psi_1, \psi_2}^{-1} \int_{A'_1 \leq |a| \leq A'_2} da \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db, \\ \forall x \in \mathbb{R}, A'_2 \geq A_2, 0 < A'_1 \leq A_1.$$

(iv) $\implies$ (v). For any  $A'_2 \geq A_2$  and  $0 < A'_1 \leq A_1$ ,

$$\left| C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right| \\ \leq |C_{\psi_1, \psi_2}| + \int_{-\infty}^{+\infty} \frac{1}{|a|} |\hat{\psi}_1(a\omega) \hat{\psi}_2(a\omega)| da \\ = |C_{\psi_1, \psi_2}| + \int_{-\infty}^{+\infty} \frac{1}{|a|} |\hat{\psi}_1(a) \hat{\psi}_2(a)| da \\ < +\infty.$$

Let  $\hat{g}(\omega) = \hat{f}(\omega) \left( C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right)$ . Then we have  $g \in L^2(\mathbb{R})$ . By Lemma 3.3, we have

$$\begin{aligned} & \langle f - f_{A'_1, A'_2}, g \rangle \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{-\infty}^{+\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left( C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right) d\omega \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \left| C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right|^2 d\omega. \quad (3.13) \end{aligned}$$

But  $f = f_{A'_1, A'_2}$ . Hence

$$\hat{f}(\omega) \left( C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right) = 0, \quad \text{a.e. } \omega \in \mathbb{R}.$$

It follows that

$$C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da = 0, \quad \text{a.e. } \omega \in \text{supp } \hat{f}.$$

Since  $\int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da$  is a continuous function on  $\mathbb{R}$  with respect to  $\omega$ , we have

$$C_{\psi_1, \psi_2} = \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da, \quad \forall \omega \in \text{supp } \hat{f}.$$

(v)  $\implies$  (vi). For any  $A'_2 \geq A_2$ ,  $0 < A'_1 \leq A_1$  and  $\omega \in \text{supp } \hat{f}$ , we have

$$\begin{aligned} & \int_{0 < |a| < A'_1 |\omega|} \frac{1}{|a|} \overline{\hat{\psi}_1(a)} \hat{\psi}_2(a) da + \int_{|a| > A'_2 |\omega|} \frac{1}{|a|} \overline{\hat{\psi}_1(a)} \hat{\psi}_2(a) da \\ &= C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \\ &= 0. \end{aligned} \quad (3.14)$$

Take derivatives with respect to  $A'_2$  on both sides, we get

$$|\omega| \left( \overline{\hat{\psi}_1(A'_2 |\omega|)} \hat{\psi}_2(A'_2 |\omega|) + \overline{\hat{\psi}_1(-A'_2 |\omega|)} \hat{\psi}_2(-A'_2 |\omega|) \right) = 0, \quad \omega \in \text{supp } \hat{f}. \quad (3.15)$$

Since (3.15) holds for every  $A'_2 \geq A_2$ , we have

$$\text{ess sup}\{|\omega| : \omega \in \text{supp } P_{\psi_1, \psi_2}\} < \infty.$$

Moreover, we also have

$$\text{ess inf}\{|\omega| : \omega \in \text{supp } \hat{f}\} > 0.$$

Otherwise,  $C_{\psi_1, \psi_2} = 0$ , which contradicts with the hypothesis. Similar arguments show that

$$\text{ess inf}\{|\omega| : \omega \in \text{supp } P_{\psi_1, \psi_2}\} > 0 \quad \text{and} \quad \text{ess sup}\{|\omega| : \omega \in \text{supp } \hat{f}\} < \infty.$$

Hence  $\hat{f}, P_{\psi_1, \psi_2} \in AC(\mathbb{R})$ .

(vi) $\implies$ (iii). Without loss of generality, we may assume that

$$\text{supp } \hat{f} \subset [-\tilde{\omega}_f, -\omega_f] \cup [\omega_f, \tilde{\omega}_f], \quad 0 < \omega_f < \tilde{\omega}_f < +\infty,$$

and

$$\text{supp } P_{\psi_1, \psi_2} \subset [-M_2, -M_1] \cup [M_1, M_2], \quad 0 < M_1 < M_2 < +\infty.$$

Let  $A_2 = M_2/\omega_f$ ,  $A_1 = M_1/\tilde{\omega}_f$ . For any  $\omega \in \text{supp } \hat{f}$ ,  $A'_2 \geq A_2$  and  $0 < A'_1 \leq A_1$ , we have

$$A'_1|\omega| \leq A_1|\omega| \leq A_1\tilde{\omega}_f = M_1, \quad A'_2|\omega| \geq A_2|\omega| \geq A_2\omega_f = M_2.$$

Hence

$$C_{\psi_1, \psi_2} = \int_{-\infty}^{+\infty} \frac{1}{|a|} \overline{\hat{\psi}_1(a)} \hat{\psi}_2(a) da = \int_0^{+\infty} \frac{1}{a} P_{\psi_1, \psi_2}(a) da = \int_{M_1}^{M_2} \frac{1}{a} P_{\psi_1, \psi_2}(a) da.$$

On the other hand, for any  $g \in L^2(\mathbb{R})$ , we see from Lemma 3.3 that

$$\begin{aligned} & \langle f - f_{A'_1, A'_2}, g \rangle \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{-\infty}^{+\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left( C_{\psi_1, \psi_2} - \int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} \overline{\hat{\psi}_1(a\omega)} \hat{\psi}_2(a\omega) da \right) d\omega \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{\omega_f \leq |\omega| \leq \tilde{\omega}_f} \hat{f}(\omega) \overline{\hat{g}(\omega)} \\ & \quad \times \left( C_{\psi_1, \psi_2} - \int_{A'_1|\omega| \leq |a| \leq A'_2|\omega|} \frac{1}{|a|} \overline{\hat{\psi}_1(a)} \hat{\psi}_2(a) da \right) d\omega \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{\omega_f \leq |\omega| \leq \tilde{\omega}_f} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left( C_{\psi_1, \psi_2} - \int_{A'_1|\omega|}^{A'_2|\omega|} \frac{1}{a} P_{\psi_1, \psi_2}(a) da \right) d\omega \\ &= \frac{1}{2\pi C_{\psi_1, \psi_2}} \int_{\omega_f \leq |\omega| \leq \tilde{\omega}_f} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left( C_{\psi_1, \psi_2} - \int_{M_1}^{M_2} \frac{1}{a} P_{\psi_1, \psi_2}(a) da \right) d\omega \\ &= 0. \end{aligned} \tag{3.16}$$

Since  $g$  is arbitrary, we have  $f(x) = f_{A'_1, A'_2}(x)$ , a.e. on  $\mathbb{R}$ . The proof of [5, Proposition 2.4.2] shows that  $f_{A'_1, A'_2}(x)$  is a continuous function on  $\mathbb{R}$  when  $f \in L^1(\mathbb{R})$ . So we have

$$f(x) = f_{A'_1, A'_2}(x), \quad \forall x \in \mathbb{R}, \forall A'_2 \geq A_2, 0 < A'_1 \leq A_1.$$

It follows that

$$\begin{aligned} & \left| (\tau(s, t)f)(x) - C_{\psi_1, \psi_2}^{-1} \int_{A'_1|s| \leq |a| \leq A'_2|s|} da \right. \\ & \quad \times \left. \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db \right| \\ &= |(\tau(s, t)f)(x) - (\tau(s, t)f_{A'_1, A'_2})(x)| \\ &= 0, \quad \forall (s, t) \in \mathcal{G}, x \in \mathbb{R}, A'_2 \geq A_2, 0 < A'_1 \leq A_1. \end{aligned}$$

(iii) $\implies$ (ii) and (ii) $\implies$ (i) are obvious. This completes the proof.  $\square$

The following is an immediate consequence.

**Corollary 3.5.** Suppose that  $f \in AC(\mathbb{R}) \cap L^1(\mathbb{R})$  is continuous, that  $\psi_1, \psi_2, x\psi_2 \in L^1(\mathbb{R})$ , that  $\psi_2$  is differentiable with  $\psi_2' \in L^2(\mathbb{R})$ , that  $\hat{\psi}_1(0) = 0 = \hat{\psi}_2(0)$ , and that  $P_{\psi_1, \psi_2} \in AC(\mathbb{R})$ . Then there exist  $A_2 > A_1 > 0$  such that

$$\begin{aligned} (\tau(s, t)f)(x) &= C_{\psi_1, \psi_2}^{-1} \int_{A_1'|s| \leq |a| \leq A_2'|s|} da \\ &\times \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db, \\ \forall x \in \mathbb{R}, (s, t) \in \mathcal{G}, A_2' \geq A_2, 0 < A_1' \leq A_1. \end{aligned} \quad (3.17)$$

The following is an explicit example.

**Example 3.1.** Let

$$\hat{\psi}(\omega) = \hat{\psi}_1(\omega) = \hat{\psi}_2(\omega) = \chi_{[1,2]} * \chi_{[1,2]} * \chi_{[1,2]}(\omega),$$

and

$$\hat{f}(\omega) = \chi_{[1/2,1]} * \chi_{[1/2,1]}(\omega).$$

Set  $A_2 = 6$  and  $A_1 = 3/2$ . Then for any  $x \in \mathbb{R}$ ,  $A_2' \geq A_2$ ,  $0 < A_1' \leq A_1$  and  $(s, t) \in \mathcal{G}$ , we have

$$\begin{aligned} (\tau(s, t)f)(x) &= C_{\psi_1, \psi_2}^{-1} \int_{A_1'|s| \leq |a| \leq A_2'|s|} da \\ &\times \int_{-\infty}^{+\infty} \frac{1}{a^2} \langle \tau(s, t)f, \tau(a, b)\psi_1 \rangle (\tau(a, b)\psi_2)(x) db. \end{aligned}$$

**Proof.** It is easy to see that

$$\int_{-\infty}^{+\infty} \chi_{[1,2]}(\omega) e^{ix\omega} d\omega = \int_1^2 e^{ix\omega} d\omega = \frac{e^{2ix} - e^{ix}}{ix}.$$

Hence  $\psi(x) = (\hat{\psi})^\wedge(-x)/2\pi = (e^{-2ix} - e^{-ix})^3/2\pi(-ix)^3$ . Obviously,  $\psi$  is differentiable and  $\psi' \in L^2(\mathbb{R})$  which follows immediately from  $(\psi')^\wedge(\omega) = i\omega\hat{\psi}(\omega) \in L^2(\mathbb{R})$ . Since  $|1 - e^{ix}| \leq |x|$ , we have

$$\int_{|x| \leq 1} \frac{|1 - e^{ix}|^3}{|x|^3} dx \leq 2.$$

On the other hand,

$$\int_{|x| > 1} \frac{|1 - e^{ix}|^3}{|x|^3} dx \leq 8 \int_{|x| > 1} \frac{1}{|x|^3} dx < +\infty.$$

Hence

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\psi(x)| dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|1 - e^{ix}|^3}{|x|^3} dx \end{aligned}$$

$$= \frac{1}{2\pi} \int_{|x| \leq 1} \frac{|1 - e^{ix}|^3}{|x|^3} dx + \frac{1}{2\pi} \int_{|x| > 1} \frac{|1 - e^{ix}|^3}{|x|^3} dx \\ < +\infty,$$

which follows that  $\psi \in L^1(\mathbb{R})$ . Similar arguments show that  $x\psi \in L^1(\mathbb{R})$ . Thus  $\psi$  satisfies the conditions in [Theorem 3.4](#). Obviously,  $f$  is a continuous function in  $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ ,  $\text{supp } \hat{f} = [1, 2]$  and  $\text{supp } \hat{\psi} = [3, 6]$ . It is easy to see that

$$C_\psi = \int_{-\infty}^{+\infty} \frac{1}{|\omega|} |\hat{\psi}(\omega)|^2 d\omega = \int_3^6 \frac{1}{\omega} |\hat{\psi}(\omega)|^2 d\omega.$$

Choose  $A_1 = 3/2$  and  $A_2 = 6$ . Then for any  $\omega \in \text{supp } \hat{f}$ ,  $A'_2 \geq A_2$  and  $0 < A'_1 \leq A_1$ , since  $A'_1 \omega \leq A_1 \omega \leq 2A_1 = 3$ ,  $A'_2 \omega \geq A_2 \omega \geq A_2 = 6$ , we have

$$\int_{A'_1 \leq |a| \leq A'_2} \frac{1}{|a|} |\hat{\psi}(a\omega)|^2 da = \int_{A'_1 \omega}^{A'_2 \omega} \frac{1}{|a|} |\hat{\psi}(a)|^2 da = \int_3^6 \frac{1}{a} |\hat{\psi}(a)|^2 da = C_\psi.$$

Now the conclusion follows from [Theorem 3.4](#).  $\square$

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